Revisiting the maximum-value test for the frequency analysis of extreme events

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Validation of probabilistic models based on goodness-of-fit tests is an essential step for the frequency analysis of extreme values. A **common drawback of these tests is that** their outcome is mainly determined by the behavior of the hypothetical model in the central part of the distribution, while **the behavior in the tails is relatively unimportant** for the results of the test. The maximum-value test is a suitable, but seldom applied, technique to overcome this problem. The test is specifically targeted to verify if the maximum (or minimum) values in the sample are consistent with the hypothesis that the distribution $F_X(x)$ is the real parent distribution. However, **usual applications of this test fail to consider the effects of parameter-estimation**: when parameters are estimated on the same sample used for verification the power of the test is relevantly decreased. **We propose here a simple, analytically explicit, technique to suitably account for this effect**, based on the application of L-moments estimators of the parameters.

1. Basic definitions

X is a sample of **n** elements distributed according to $\mathbf{F_X(x)}$. The quantile function of $\mathbf{F_X(x)}$ is given by

$$x(F) = \theta_1 + \theta_2 \cdot z(F, \theta_3) \qquad \text{(a)}$$

where θ_1 = position parameter; θ_2 = scale parameter; θ_3 =shape parameter; **z** = standardized variable

The Maximum Value Test requires that

$$\left[F_X\left(X_{(n)} \mid \Theta\right)\right]^n < 1 - \alpha \qquad \text{(b)}$$

where $\mathbf{X_{(n)}}$ is the sample maximum value, Θ is the estimated parameter set and α is the significance level of the test.

2. Standard parameter estimation

Standard L-moments can be expressed as

$$\begin{cases} L_1 = \theta_1 + \theta_2 A(\theta_3) \\ L_2 = \theta_2 B(\theta_3) \\ L_3 = \theta_2 C(\theta_3) \end{cases} \qquad \text{(c)}$$

Parameters estimators are obtained by equating (c) with the sample L-mom. For 2-parameter distributions one finds

$$\hat{\theta}_2 = \frac{l_2}{B(\hat{\theta}_3)} \rightarrow \hat{\theta}_2 = \frac{l_2}{\ln(2)}$$

$$\hat{\theta}_1 = l_1 - \hat{\theta}_2 A(\hat{\theta}_3) \rightarrow \hat{\theta}_1 = l_1 - \gamma_E \hat{\theta}_2$$

Gumbel

3. Modified L-Moments

$\mathbf{X_{(n)}}$ is replaced by the median of the max value distribution

$$\left[F\left(\hat{X}_{(n)}\right)\right]^n = 0.5 \rightarrow \hat{X}_{(n)} = \theta_1 + \theta_2 \cdot f(n, \theta_3) \qquad \text{(d)}$$

Observe that sample L-moments can be written as

$$l_1 = \frac{1}{n}\sum_{i=1}^{n-1} X_{(i)} + \frac{1}{n}\hat{X}_{(n)} = l_1' + \frac{1}{n}\hat{X}_{(n)}$$

$$l_2 = \frac{1}{n}\sum_{i=1}^{n-1}\left(\frac{2(i-1)}{n-1} - 1\right)X_{(i)} + \frac{1}{n}\hat{X}_{(n)} = l_2' + \frac{1}{n}\hat{X}_{(n)} \qquad \text{(e)}$$

$$l_3 = \frac{1}{n}\sum_{i=1}^{n-1}\left(\frac{6(i-1)(i-2)}{(n-1)(n-2)} - \left(\frac{6(i-1)}{(n-1)}\right) + 1\right)X_{(i)} + \frac{1}{n}\hat{X}_{(n)} = l_3' + \frac{1}{n}\hat{X}_{(n)}$$

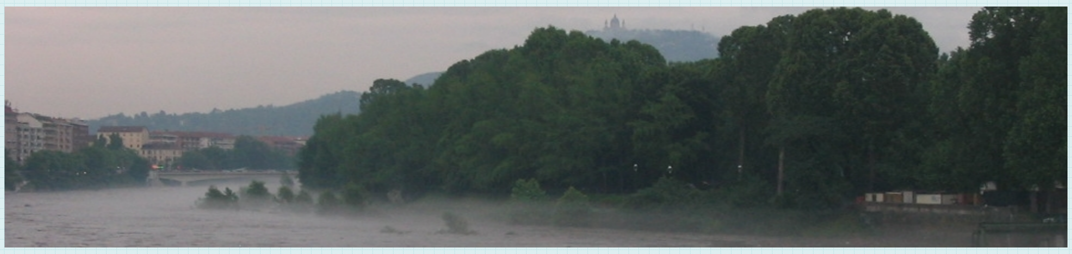
One can find the new estimators for θ_1, θ_2 and θ_3 (which are distribution dependent) by equating (e) and (c)

For example for the Gumbel distribution one finds:

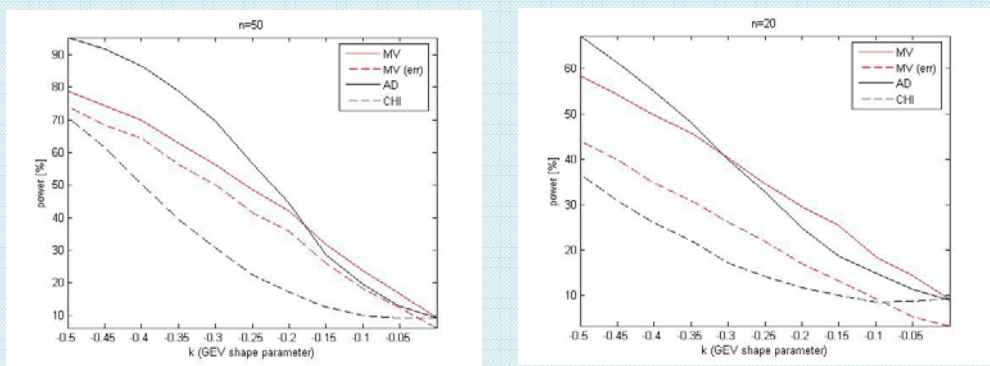
$$\hat{\theta}_2 = \frac{l_2'(1 - 1/n) + l_1'/n}{\ln(2)(1 - 1/n) + \gamma_E/n - \hat{f}/n}; \quad \hat{\theta}_1 = l_1' - l_2' - \theta_2[\gamma_E - \ln(2)]; \quad \hat{f} = -\ln[\ln(2)/n]$$

4. Application

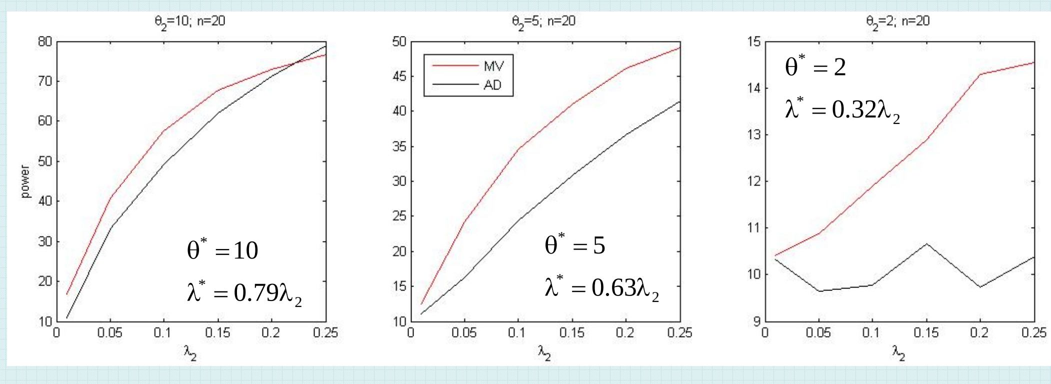
Generate TCEV- and GEV-distributed samples and apply the Maximum Value (MV), Chi-Squared (CHI) and Anderson-Darling (AD) test under the hypothesis $\mathbf{H_0}$: $\mathbf{F_X(x)}$ **= Gumbel distribution**. The power of the tests is evaluated for different parameter sets and sample numerosities.



Power of the tests (as indicated in legends) for <u>GEV samples</u>:



and <u>TCEV samples</u>: $G_X(x) = \exp\left\{-\lambda_1 e^{-x/\theta_1} - \lambda_2 e^{-x/\theta_2}\right\}$



5. Discussion

- The MV test performs much better than the Chi-squared and slightly better than the AD test when the parent distribution is similar to the distribution considered in the null hypothesis;
- Since the test considers only the maximum value, it is particularly suited when small samples are available, with the advantage of being simple and analytically explicit.